Received: *August 28, 2000*, Accepted: *November 20, 2000*
Revised: *October 12, 2000*

Decay modes of highly excited string states and Kerr black holes

Roberto Iengo and Jussi Kalkkinen*

SISSA, via Beirut 4, Trieste 34014, Italy
INFN, Sezione di Trieste
E-mail: iengo@sissa.it, kalkkine@sissa.it

Abstract: We consider four-graviton scattering in type- II string theory on one-loop level in the large centre of mass energy M^2 limit. We extract from it an explicit integral expression for the full string theory corrections to the imaginary part of the mass-shift and the lifetime of a massive state with the highest allowed spin $J = 2M^2 + 2$. We find a decay rate that is up to $\log M$ corrections of order 1 in string units, times g_s^2. We also find that the dominant decay mode corresponds to the emission of light particles, whereas the decay into two massive or two massless states is exponentially suppressed. We discuss the relation of our results to quantum gravity aspects of a Kerr black hole.

Keywords: Superstrings and Heterotic Strings, Black Holes in String Theory.

*This work was supported by the European Union TMR program CT960045.

Contents

1. Introduction

String theory provides a very non-trivial regularization of ultraviolet phenomena in gravity. One of the many surprising features of this regularization is the fact that increasing the mass of a state does not necessarily lead to decreasing its lifetime; string theory dynamics seems to know how to suppress the kinematically allowed phase space crucially. We calculate the decay width and the mass-shift of a massive string state that displays this behaviour. As we are far outside the validity of a field theory analysis we have to consider all string theory corrections in the calculation at the same time, in order to see whether it really persists; the field theory limit is not enough.

We will perform the evaluation of the lifetime by means of a direct computation: we consider the one-loop contribution to the mass-shift of the massive state and evaluate its imaginary part in the type-IIA or -IIB superstring theory. We will obtain the relevant amplitude by starting from the well-known formula for the one-loop four graviton scattering amplitude in superstring theory, and extracting from it the residue of the singularity when the Mandelstam variable s approaches the mass of the massive state. We have studied the case of the (unique, non degenerate) maximal angular momentum state, which we can select by looking at the dependence on the scattering angle, fixing the normalization by comparing it with the residue at the pole of the same four graviton amplitude at tree-level. Since we work at one-loop level, we are considering the lifetime as the massive state is allowed to decay into two states only; this is the only possibility at the first non-trivial order in the perturbative expansion.

The analysis is rather subtle, because the resulting amplitude is expressed as an integral over tori: this boils down to integrating over a four real dimensional manifold that consists of both the torus surface and the fundamental domain of the complex modular parameter of the torus. This integral is formally divergent and, as usual in string theory, it must be given a meaning by analytic continuation. It is precisely this feature which gives rise to an imaginary part, as otherwise the result would be real, as the integrand is real positive. This is also very similar to what happens in standard Feynman loops, for instance in a convenient exponential parametric representation of the propagators: the resulting integral is convergent in the euclidean domain, when the Mandelstam variable s is space-like, and it is formally divergent if s is time-like, which is relevant for the decay. The correct result is obtained by analytic continuation. In the case of string theory one cannot start from a naive off-shell configuration, as there are consistency requirements, such as preserving modular invariance, but the strategy is similar. We have first checked the method by computing the lifetime for the decay of the massive state into two massless string modes, which can be compared — and it agrees — with a direct computation of the decay into two gravitons. This can be done by extracting the tree-level coupling of the massive state to two gravitons from the tree-level four graviton amplitude. The result is that this decay channel is exponentially suppressed for our massive state (it goes like $\sim \exp -\beta M^2$ for $\beta > 0$ a numerical constant).

The full investigation, including any two body channel, is based on the general method of saddle point analysis, although we also have to include contributions from more singular loci than saddle points in the integral: it turns out that the dominant behaviour arises in regions where the integrand has a conical singularity. Visualizing the behaviour of the integrand on sections of the manifold has required some numerical work. Happily, the numerical analysis confirms that the relevant configurations always correspond to points having particular geometrical and/or symmetrical meaning. In fact, it turns out that the number of relevant configurations is rather limited and that we could reliably follow the evolution of the integrand on them.

The final result is that the lifetime of our massive state is of order 1, up to logarithmic corrections. It is interesting that also a physical picture emerges from our analysis: the dominant decay mode is seen to be rather asymmetric, corresponding to a radiative-like process, where the very massive state decays into an other very massive state emitting a light particle. The other configurations, where the massive state splits into two states, which are both massive or both light, are exponentially suppressed.

In [1] the behaviour of the four graviton fixed angle scattering was investigated. Their results are in qualitative agreement with ours in that they, too, find exponentially suppressed decay widths for decays into light particles, gravitons in their case. Also there the dominant contributions arise in geometrically significant points. In ref. [2] the decays of open string excitations of D-branes were studied, for polarizations either parallel or transverse to the brane; for transverse polarizations the leading decays also arise from the emission of soft light particles. Ref. [3] deals with a similar problem to ours, though finding a somewhat different result. Related issues were considered in ref. [4].

The fact that the dominant decay mode is the emission of massless particles is reminiscent of Hawking radiation. Also, it is expected that the collision of gravitons at very high energy may lead to the formation of a gravitationally collapsed intermediate state [5, 6, 7]. We are thus lead to compare our results to black hole physics. The distinctive features of our state are its high angular momentum and the fact that it is in the range of validity of perturbative string theory. Therefore it should, somehow, correspond to an extrapolation of a Kerr black hole to the small string coupling region. However, the radiation rate we obtain is slightly higher than that of the Hawking radiation of a black hole. Indeed, the detailed analysis of section 5 indicates that our state also extends beyond the horizon radius of the corresponding classical Kerr solution, when the string coupling is in the perturbative region. Requiring its characteristic length scale to be of the same order as the classical horizon turns out to be equivalent to requiring that the string coupling constant have the same — large — value as what results from the Correspondence Principle [8] (see also [9]–[12]). We are thus lead to expect that the extrapolation of a classical Kerr black hole to the quantum perturbative regime corresponds to a very massive string state of large angular momentum.

2. Formula for the lifetime

We start by considering the one-loop four-graviton amplitude in the type-IIA/B superstring theory

$$A_1 = R^4 \int \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} \int \prod_{i=1}^{3} \frac{d^2 z_i}{\mathrm{Im}\,\tau} \mathrm{e}^{-2\sum k_i \cdot k_j P(z_{ij})}\,, \tag{2.1}$$

where R^4 is a kinematical factor constructed from the tensor t_8 [13] that contains the graviton polarizations. z_i are the puncture insertion points of the graviton vertices, with $z_{ij} = z_i - z_j$ and $z_4 = 0$, to be integrated over the torus surface. τ is the torus modulus to be integrated over the fundamental domain. $P(z_{ij})$ is the propagator of the world-sheet scalar

$$P(z) = \ln \left| \frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)} \right|^2 - 2\pi \frac{(\operatorname{Im} z)^2}{\operatorname{Im} \tau} . \tag{2.2}$$

For the entering graviton momenta k_i the Mandelstam invariants are $s = 2k_1 \cdot k_2 = 4E^2$, $t = 2k_1 \cdot k_3 = -2E^2(1 - \cos\theta)$, $u = 2k_1 \cdot k_4 = -2E^2(1 + \cos\theta)$ where E and θ are the graviton energy and the scattering angle in the CM frame. We work in units of α'.

The amplitude A_1 has a double pole for $s \to N$ due to the propagator of a massive string state with $M^2 = N$, produced by the collision of the two incoming gravitons entering into the loop (i.e. the torus), and another similar propagator emerging from the loop and decaying into the outgoing gravitons. The residue of this double pole is proportional to ΔM^2, the mass-shift of the massive state: $M^2 \to M^2 + \Delta M^2$. The imaginary part of ΔM^2 gives the inverse lifetime of the state.

In order to get ΔM^2 we divide the double pole residue of A_1 by the residue of the simple pole (for $s \to N$) of A_0, the tree-level four-graviton scattering amplitude. In fact,

$$A_1 \longrightarrow G_{\text{in}} \frac{1}{s-N} \Delta M^2 \frac{1}{s-N} G_{\text{out}} , \tag{2.3}$$

where $G_{\text{in(out)}}$ represent the coupling of the state to the incoming (resp. outgoing) gravitons, and

$$A_0 \longrightarrow G_{\text{in}} \frac{1}{s-N} G_{\text{out}} . \tag{2.4}$$

The poles of A_1 for $s \to N$ occur as singularities of the integral over z_i for $z_{12} \to 0$ and $z_{34} \to 0$. The integrand behaves as

$$\sim |z_{12}|^{-2s} |z_{34}|^{-2s} \cdot F(z_{12}, z_{34}, z) , \tag{2.5}$$

where $z = (z_1 + z_3)/2$. In order to get the poles we have to look at the terms of the expansion of the remaining function F that behave as $|z_{12}|^{2(N-1)} |z_{34}|^{2(N-1)}$. Let us now consider the string state with maximal angular momentum $J = 2N + 2$ and $M^2 = N$, and thus look for the maximal power of $\cos\theta$, or else of $(t - u)$, in the residue. We find a unique term of that kind in the expansion of F, namely

$$\begin{aligned} F(z_{12}, z_{34}, z) \sim (E^2 \cos\theta)^4 (\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4) \left(\frac{t-u}{2} \right)^{2N-2} \frac{1}{(2N-2)!} \mathrm{e}^{2NP(z)} \times \\ \times |z_{12}|^{-2N} |z_{34}|^{-2N} \left(\left(f''(z) + \frac{\pi}{\operatorname{Im}\tau} \right) z_{12} z_{34} + \right. \\ \left. + \left(\bar{f}''(z) + \frac{\pi}{\operatorname{Im}\tau} \right) \bar{z}_{12} \bar{z}_{34} - \frac{\pi}{\operatorname{Im}\tau} (z_{12} \bar{z}_{34} + \bar{z}_{12} z_{34}) \right)^{2N-2} , \end{aligned} \tag{2.6}$$

where the term $(E^2 \cos\theta)^4 (\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4)$ arises from R^4, ϵ_i being the graviton polarization tensor, and $f'' = \partial_z^2 \theta(z|\tau)$.

The residue of the double pole, for the maximal J, turns out to be

$$\begin{aligned}\text{Res } A_1 \sim (E^2\cos\theta)^4(\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4)\frac{1}{(2N-2)!}\left(\frac{t-u}{2}\right)^{2N-2}\times \\ \times\int\frac{d^2\tau}{(\operatorname{Im}\tau)^2}\int\frac{d^2z}{\operatorname{Im}\tau}\mathrm{e}^{2NP(z)}\sum_{n=0}^{N-1}\frac{(2N-2)!}{(n!)^2(N-n-1)!^2}\times \\ \times\left|f^{``}(z)+\frac{\pi}{\operatorname{Im}\tau}\right|^{2n}\left(\frac{\pi}{\operatorname{Im}\tau}\right)^{2N-2n}.\end{aligned} \tag{2.7}$$

On the other hand, the residue at the pole for $s\to N$ of the tree-level amplitude for the maximal J is obtained from the Veneziano amplitude for the closed superstring

$$A_0 = R^4\frac{\Gamma(-s)\Gamma(-t)\Gamma(-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)} \tag{2.8}$$

and it is

$$\text{Res } A_0 \sim \left(\frac{t-u}{2}\right)^{2N-2}\frac{1}{(N!)^2}(E^2\cos\theta)^4(\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4)\,. \tag{2.9}$$

Finally, using a standard formula for the Legendre polynomial,

$$\begin{aligned}\Delta M^2 \sim N^2\int\frac{d^2\tau}{(\operatorname{Im}\tau)^2}\int\frac{d^2z}{\operatorname{Im}\tau}\mathrm{e}^{2NP(z)}\left(\frac{\pi}{\operatorname{Im}\tau}\right)^{2N}\times \\ \times(Q-1)^{N-1}\mathrm{P}_{N-1}\left(\frac{Q+1}{Q-1}\right),\end{aligned} \tag{2.10}$$

where P_n is the Legendre polynomial of degree n and $Q = |f^{``}(z)\operatorname{Im}\tau/\pi+1|^2$. The sign $\sim$ indicates here, and elsewhere in this article, that we have left out an N-independent normalization. As in this expression the combinations Q and $\pi\exp(P)/\operatorname{Im}\tau$ are separately modular invariant the mass-shift ΔM^2 is well defined.

The inverse lifetime of the massive state is given by the imaginary part of ΔM^2, divided by $M=\sqrt{N}$

$$\Gamma = \frac{\operatorname{Im}\Delta M^2}{\sqrt{N}}\,. \tag{2.11}$$

Note that ΔM^2 is expressed, formally, as an integral of a real quantity, but this integral is actually divergent. The imaginary part comes from the fact that this expression has a meaning in the sense of an analytic continuation.

3. Decay rate into gravitons

As a first investigation of the Physics of the decay of the massive string mode we compute its lifetime for decaying into two gravitons. This computation can be done in two ways:

First, by extracting the coupling of the massive state to the gravitons by looking at the pole of the tree-level amplitude A_0 and then performing the integral over the phase space, or,

Second, by extracting the contribution of the two graviton channel from the formula for the inverse lifetime we derived, starting by the double pole residue of the one-loop amplitude A_1. This can be done by looking at the dominant contribution in the pinching limit $\operatorname{Im}\tau \to \infty$ of the torus, corresponding to massless states circulating into the loop.

In the first strategy we start by recalling that

$$\text{Res } A_0 \sim (2p^2 \cos\theta)^J \frac{1}{(N!)^2} (\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4)\,, \tag{3.1}$$

where the CM square space momentum is $p^2 = s/4$. As said, $J = 2N + 2$ and we have taken the maximal power of $\cos\theta$. On the other hand, the Feynman graph describing two incoming gravitons forming a massive state of angular momentum J further emitting two outgoing gravitons would be (looking at the maximal power of $\cos\theta$)

$$\begin{aligned} A_0 &\sim (\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4) \frac{g_J^2}{s - M^2} \sum_V p^{r_1} \cdots p^{r_J} V_{r_1 \cdots r_J} V_{s_1 \cdots s_J} p'^{s_1} \cdots p'^{s_J} \\ &\sim (\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4) \frac{g_J^2}{s - M^2} (p^2 \cos\theta)^J\,, \end{aligned} \tag{3.2}$$

where $V_{r_1 \cdots r_J}$ is the (symmetric and traceless) polarization tensor of the massive state, and the indices r_i run over the $d = 9$ space dimensions. By comparing the pole residue we get that (up to an N-independent coupling constant):

$$g_J^2 = \frac{2^{2N+2}}{(N!)^2}\,. \tag{3.3}$$

Now we can directly compute the inverse lifetime for the decay into two gravitons. It is (up to N-independent numerical constants)

$$\begin{aligned} \Gamma_{2\text{grav}} &\sim \frac{1}{\sqrt{N}} g_J^2 \int \frac{d^d p}{p^2} \delta\left(p - \frac{\sqrt{s}}{2}\right) V_{r_1 \cdots r_J} p^{r_1} \cdots p^{r_J} p^{s_1} \cdots p^{s_J} V_{s_1 \cdots s_J} \\ &\sim \frac{1}{\sqrt{N}} g_J^2 p^{d-3} p^{2J} J^{-\frac{d-1}{2}} \frac{2^J J!^2}{(2J)!}\,. \end{aligned} \tag{3.4}$$

We have assumed the normalization $||V|| = 1$ and made use of the formula

$$\int d^d \hat{p} \delta(\hat{p}^2 - 1) V_{r_1 \cdots r_J} \hat{p}^{r_1} \cdots \hat{p}^{r_J} \hat{p}^{s_1} \cdots \hat{p}^{s_J} W_{s_1 \cdots s_J} \sim V \cdot W J^{-\frac{d-1}{2}} \frac{2^J J!^2}{(2J)!}\,, \tag{3.5}$$

which holds for symmetric and traceless V and W. The angular integration for large J gives the factor $J^{-\frac{d-1}{2}}$.

Notice that the dependence on the dimensionality d disappears in the final result. By using the Stirling formula for large N and $J = 2N+2$ and putting $p^2 = N/4$, we get that the inverse lifetime for the decay into two gravitons is exponentially small:

$$\Gamma_{2\mathrm{grav}} \sim \left(\frac{\mathrm{e}}{4}\right)^{2N} . \tag{3.6}$$

This is already a surprising result, because one would have expected the massive state to decay very rapidly due to the large phase-space. Instead, the indication is that the very high mass, very high angular momentum string states might have a long lifetime. We will see that that this is indeed true also when summing over all the two-body channels: the decay into two states of comparable mass is in general exponentially suppressed, although the dominant contribution coming from the decay into a massless and another very massive state is of order unity $\sim N^0$, times logarithmic corrections.

For the moment we continue with the computation of the lifetime for decay into two massless particles by using now the second method, based on the one-loop (torus) amplitude and looking at the pinching limit of the torus.

We write $\tau = \tau_1 + i\tau_2$ and $z = x + y\tau$. Thus x, y vary between 0 and 1 whereas τ spans the fundamental domain. The relevant pinching limit corresponds to $\tau_2 \to \infty$ with $y \neq 0, 1$. In this limit we have, putting $y = 1/2 + \eta$,

$$\mathrm{e}^{2NP(z)} \longrightarrow \frac{\mathrm{e}^{N\pi\tau_2}}{(4\pi^2)^{2N}} \mathrm{e}^{-4N\pi\tau_2\eta^2} . \tag{3.7}$$

Further, in this limit $Q \to 1$ and we get

$$\lim_{Q\to 1} (Q-1)^{N-1} P_{N-1}\left(\frac{Q+1}{Q-1}\right) = \frac{(2N-2)!}{(N-1)!^2} . \tag{3.8}$$

In order to keep only the contribution of the two-massless particle channel, we replace the integrand with its pinching limit expression and get

$$\Gamma_{2\mathrm{massless}} \sim N^{3/2} \frac{(2N-2)!}{(N-1)!^2} \,\mathrm{Im} \int \frac{d^2\tau}{\tau_2^5} \frac{\mathrm{e}^{N\pi\tau_2}}{(4\pi^2)^{2N}} \left(\frac{\pi}{\tau_2}\right)^{2N-2} \int d\eta \tau_2 \mathrm{e}^{-4N\pi\tau_2\eta^2} . \tag{3.9}$$

After doing the gaussian integral in η, we have to get the imaginary part of the integral over τ, which is expected from the integral over τ_2 extending up to infinity and formally divergent. We have thus to consider

$$\mathrm{Im} \int d\tau_2 \tau_2^{-2N-2-1/2} \mathrm{e}^{N\pi\tau_2} \sim \frac{(N\pi)^{2N+1+1/2}}{\Gamma(2N+2+1/2)} . \tag{3.10}$$

This result is obtained by analytic continuation. A quick way of getting it is to look for the saddle point of the integrand: we find a minimum at $\tau_2 = (4/\pi)^2$ and thus expanding around it we get an inverted gaussian which integrates to an imaginary result. By putting the various factors together we finally get

$$\Gamma_{2\text{massless}} \sim \left(\frac{\mathrm{e}}{4}\right)^{2N}, \tag{3.11}$$

that is, up to a numerical factor, the same as $\Gamma_{2\text{grav}}$ computed directly by Feynman rules. Evidently the other possible massless channels beside the two gravitons give the same large-N behaviour.

4. Decay rate into string states

We shall now calculate the lifetime of the massive state when it is allowed to decay also to massive string excitations in addition to the massless gravitons considered above. This amounts, in principle, to performing the integral (2.10) over the full fundamental domain and the torus.

4.1 Saddle point analysis

As we are only interested in the large-N limit of the amplitude, the leading contributions should arise, at least if the integral were convergent, from where the integrand reaches its maximum. In the present case we shall have to deal with the difficulty that the integral is actually divergent: as we saw above, we could regularize it by analytically continuing one of the variables of integration into the complex plane; The correct leading contribution then came from near the point where the original integrand reached its minimum.

In this context, saddle point analysis is actually not just one of the many ways to regularize a divergent integral. The amplitudes here, as they are in field theory in general, are generically divergent: what actually is physically significant, is the behaviour of the integrals near the extrema of the integrands after an analytic continuation (Wick rotation) in the variables of integration. Hence, we should find the extrema of the integrand and to consider contributions coming from a gaussian analysis near them. The integration contour along the parameters of the unstable directions should then be deformed in the complex plane to coincide with the imaginary axis in a local neighbourhood of the saddle point, so that the gaussian integral could finally be calculated.

We should also consider separately all the singularities, orbifold points, and any other special points that one might run into in the integration region, as they will play the role of end points of integration. In our case, the integration manifold has no true boundaries, since it is described by two complex variables: the coordinate z over the torus surface, which has no boundary, and the torus modulus parameter τ, which runs over the fundamental domain folded into itself by modular transformations (we will recall below the most relevant facts). For the convergent case the reason for this is clear: one is supposed to calculate the contribution coming from these points and simply to find out if it might be the leading one or not. For the divergent case it is useful to think of these special points as kind of topological defects in the integration range.

The formula (2.10) for the mass-shift can be conveniently cast in the form

$$\Delta M^2 \sim N^2 \int \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} \int \frac{d^2 z}{\mathrm{Im}\,\tau} \left(\mathrm{e}^{P(z)} \frac{\pi}{\mathrm{Im}\,\tau}\right)^2 t(z|\tau)^{2N-2} g_N(\sqrt{Q})\,, \tag{4.1}$$

where

$$\begin{aligned} t(z|\tau) &= \mathrm{e}^{P(z)} \frac{\pi}{\mathrm{Im}\,\tau}(\sqrt{Q}+1)\,, \\ g_N(x) &= \left(\frac{x-1}{x+1}\right)^{N-1} \mathrm{P}_{N-1}\left(\frac{x^2+1}{x^2-1}\right). \end{aligned} \tag{4.2}$$

The function g_N behaves very mildly in the large-N limit: on the positive real axis it is bounded between 1 and $1/\sqrt{\pi(N-1)}$; in the large-N limit, for fixed $Q \neq 0$, it actually approaches [14] the value $\sim 1/\sqrt{N}$ except for Q very near to 0. By using the integral representation of the Legendre polynomial, it is seen that this also holds for Q growing as a power of $\log N$ (up to $\log N$ corrections). We will see that this is the region where we get the maximal contribution. In our analysis we can hence safely approximate it with this value, and concentrate below on the function $t(z|\tau)$. The integral behaves therefore at $N \to \infty$ as

$$\Delta M^2 \sim \frac{N^{3/2}}{\sqrt{4\pi}} \int \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} \int \frac{d^2 z}{\mathrm{Im}\,\tau} (Q^{3/4} + Q^{1/4})^{-1} t(z|\tau)^{2N}\,. \tag{4.3}$$

4.2 Critical points

We can expect *a priori* that, due to the symmetry of the problem, the dominant contribution to the integral and, in particular, to its imaginary part comes from the neighbourhood of points that have a geometrical meaning. We will see that this is indeed the case. Also, it is important to keep in mind the physical meaning of the coordinate z: the double pole residue of the four graviton torus amplitude represents the amplitude of two vertices (of the massive state) inserted on the torus surface at the point z and at the origin, respectively.

Let us review some main features. The integrand t, and also Q as well, are symmetric for $z \to -z$ and their first derivatives in z vanish for the points $z = 1/2$, $z = \tau/2$ and $z = 1/2 + \tau/2$. These points are related using the modular transformations

$$\begin{aligned} T &: \tau \longmapsto -\frac{1}{\tau} \\ S &: \tau \longmapsto \tau + 1 \end{aligned} \tag{4.4}$$

in such away that T interchanges $z = 1/2$ and $z = \tau/2$ but maps $z = 1/2+\tau/2$ back to itself, whereas S interchanges $z = 1/2 + \tau/2$ and $z = \tau/2$ but maps $z = 1/2$ back to itself. The point $z = 0$ is invariant under the modular transformations.

Both t and Q are modular invariant and the complete integrand, including the measure, is also modular invariant. Therefore the integration over the modulus τ is restricted as usual to the fundamental domain: $|\tau_1| \leq 1/2$ *and* $|\tau| \geq 1$. From now on we denote $\tau_1 \equiv \mathrm{Re}\,\tau$ and $\tau_2 \equiv \mathrm{Im}\,\tau$.

Modular transformations relate, in general, points outside the fundamental domain to points inside it, but the two lines $\tau_1 = \pm 1/2$ are identified by the transformation S, whereas the border segment $|\tau| = 1,\ \tau_1 > 0$ is identified with $|\tau| = 1,\ \tau_1 < 0$ by the transformation T. Thus there are two orbifold points of the fundamental domain that are mapped back to themselves under modular transformations: these are $\tau = i$ and $\tau = 1/2+i\sqrt{3}/2$. This last point is invariant under the full modular group and, there, all the three points $z = 1/2$, $z = \tau/2$ and $z = 1/2 + \tau/2$ are identified.

Of course, another special point is the pinching limit of the torus $\tau \to \infty$. In this limit the behaviour of the integrand depends on whether z remains fixed or not:

(1) For $z = 1/2$, we have $t \to 1$ and $Q \to \infty$. In this case the integral is convergent.

(2) For both $z = 1/2 + \tau/2$ and $\tau/2$ we have

$$t \longrightarrow \mathrm{e}^{\pi\tau_2/2}\frac{2\pi}{4\pi^2\tau_2} \tag{4.5}$$

and $Q \to 1$. In this case the integral is divergent.

Since the integrand is real positive, we can only expect an imaginary part from the fact that the integral, being formally divergent, has to be defined by analytic continuation. One would thus be tempted to exclude the point $z = 1/2$, but since it is related to the other points by modular transformations it, too, can — and will — play a role.

As for the point $z = 0$, there we have $t = 1$, for any τ, and the integral is convergent, and thus it does not give rise to an imaginary part. This point therefore contributes only a power dependence in N, and only to the real part. Indeed it represents the pinching limit where the relative distance of two vertices of the massive state coalesce: it corresponds to a tadpole correction to the massive propagator from which no imaginary part is expected.

It will be convenient to analyse the integral by choosing to perform it over some geometrically meaningful modular invariant variable. A good part of the investigation has been carried out numerically, by using the software *Mathematica* to visualize the shape of the integrand on various one- and two-dimensional sections. Actually this numerical insight confirms the expectation that the relevant contribution comes from the above discussed critical points.

4.3 Modular invariant analysis

Let us recall that $Q = (|w| \operatorname{Im} \tau / \pi)^2$, where

$$w = \partial_z^2 \theta_1(z|\tau) + \frac{\pi}{\operatorname{Im} \tau}, \tag{4.6}$$

is a modular invariant quantity. In order to reveal its geometrical meaning we observe that

$$w(z) = \wp(z) + 2\zeta\left(\frac{1}{2}\right) - \frac{\pi}{\operatorname{Im}(\tau)}, \tag{4.7}$$

where $\wp$ and ζ are the respective Weierstrass functions. It is instructive to consider the integration over the complex variable w instead of z. Then the measure becomes

$$d^2 z = \frac{dw}{u} \wedge \frac{d\bar{w}}{\bar{u}}, \tag{4.8}$$

where w and $u = \wp'(z)$ satisfy the equation of the torus

$$u^2 = w^3 + g_2(\tau) w + g_3(\tau) \tag{4.9}$$

in $[w, u, 1] \in \mathbb{C}P^2$. When embedding the torus in the complex projective space, the new coordinate w is therefore one of the projective coordinates, and the new integration measure is still the standard volume measure. This change of coordinates is invertible except exactly at the branching points $z = 1/2, \tau/2$ and $1/2 + \tau/2$ where $\wp'$ vanishes.

The exponential divergence of the integrand of eq. (4.3) for $\tau_2 \to \infty$ appears only for $Q = 1$. This is because, putting $z = x + y\tau$, the integrand diverges only for $y \neq 0$ and in this case $\partial_z^2 \theta_1(z|\tau) \to 0$. It might therefore be useful to consider the integral for a fixed value of Q, i.e. integrating over Q last (Q can take all the values from 0 to ∞). As far as the imaginary part of the mass shift is concerned, we can write the integral (4.3) in the form

$$\Delta M^2 \sim N^{3/2} \int dQ \left(\int \frac{d^2\tau}{(\operatorname{Im} \tau)^3} \int d\varphi t^{2N} \right), \tag{4.10}$$

where $\arg w = 2\pi\varphi$. This rearrangement is allowed because Q is a modular invariant quantity. φ is not invariant, instead, but gets shifted in modular transformations. We observe that $\varphi \to 0$ for $\tau_2 \to \infty$.

Let us briefly state our procedure:

(1) We shall first identify where the dominant contributions arise for fixed Q and τ_2.

(2) We note that the imaginary unit appears, again, out of the integrations over τ_2.

(3) We find the imaginary part for all values of Q separately, and notice that, for different ranges of Q, the dominant contribution comes from different loci of the torus, thus corresponding to different physical processes.

(4) Finally, we notice that among these contributions the dominant one arises in a particular limit $\tau_2 \to \infty$, and we deal, in detail, with the subtleties of taking the limit.

We are now in the position to go into more detail:

(1) We find the dominant contribution for fixed Q and fixed τ_2. This amounts to finding the maximum of the integrand t as a function of τ_1 and φ. For large values of τ_2 and $\sqrt{Q} < 1.6259$ ($\sqrt{Q} > 1.6259$) this always appears at $\varphi = 0$ (resp. at $\varphi = \pm 1/2$). Decreasing τ_2 this maximum decreases monotonically, until, finally, another maximum in the (τ_1, φ)-plane takes over.

(2) When $\sqrt{Q} < 1.6259$ this dominant maximum on the (τ_1, φ)-plane turns out to correspond to the point $z = 1/2 + \tau/2$ on the torus. This maximum is actually a conical singularity: the first derivatives in τ_1 and φ are discontinuous and do not vanish in this point. Following this maximum as a function of τ_2 (for fixed Q), one finds that the value of the function fixed in the point corresponding to a *maximum* in τ_1 and φ, goes through a *minimum* in τ_2 at the boundary of the fundamental domain $|\tau| = 1$. The fact that we found a minimum as a function of τ_2 relies on the fact that when we move over the boundary of the fundamental domain at $|\tau| = 1$ we can map back to the fundamental domain using the modular transformation $T : \tau \mapsto -1/\tau$ that maps the point $z = 1/2 + \tau/2$ back to itself.

This minimum is smooth: the first derivative in τ_2 is zero, and we can therefore evaluate the inverted gaussian integral that produces the imaginary part. We have thus found a "saddle point" contribution (for fixed Q) to the imaginary part of the mass-shift. This point is actually a conical maximum on the (τ_1, φ)-plane: integrating over $\delta\tau_1$, $\delta\varphi$ and $\delta\tau_2$ around it would therefore contribute a factor proportional to $iN^{-5/2}$. In this region the integrand t is less than 1, corresponding therefore to an imaginary part exponentially suppressed for large $M^2 = N$.

(3) This procedure can be performed for all values of Q separately, and we have to pick the dominant contribution. When decreasing Q, the integrand t at the saddle point decreases, and reaches its minimum at $Q = 0$, where it corresponds to $\tau = i$, which is a fixed orbifold point for $z = 1/2 + \tau/2$. When increasing Q the value of the integrand t at the saddle point increases. For $\sqrt{Q} = 1.6259$ the location of the saddle point in the τ-plane reaches the corner of the fundamental domain $\tau = 1/2 \pm i\sqrt{3}/2$ where the three points on the torus $z = 1/2 + \tau/2$, $z = \tau/2$ and $z = 1/2$ are identified by modular transformations. When $\sqrt{Q}$ is increased above 1.6259, this saddle point (corresponding to $z = 1/2 + \tau/2$) moves out of the fundamental domain (i.e. $|\tau_1| > 1/2$), and local maxima —

still conical singularities on the (τ_1, φ)-plane — corresponding to the points $z = 1/2$ and $z = \tau/2$ move in. These points are interchanged under the modular transformation T.

There is no minimum in τ_2 inside the fundamental domain for the conical point corresponding to $z = \tau/2$, whereas there is a smooth minimum in τ_2 inside the fundamental domain for $z = 1/2$. This minimum occurs for $|\tau_1| = 1/2$. Thus, when $\sqrt{Q}$ passes through 1.6259, the "saddle point" corresponding to $z = 1/2 + \tau/2$ is replaced continuously with the "saddle point" corresponding to $z = 1/2$. We still get a factor $iN^{-5/2}$ from integrating over the variations around it, but the integrand t increases with increasing Q and ultimately $t \to 1$ for $Q \to \infty$. This is easy to check analytically, since for $\tau_2 \to \infty$ we have that, for $z = 1/2$, $\sqrt{Q}$ is proportional to τ_2 and $\theta_1(1/2|\tau)/\theta_1'(0|\tau) \to 1/\pi$.

(4) The shape of the integrand flattens in the limit as $t \sim 1 - O(\mathrm{e}^{-2\pi\tau_2})$. Therefore t^{2N} goes down exponentially with N until τ_2 reaches the region $\tau_2 \sim \log N/2\pi$, where t^{2N} begins to be of order 1. Since the integral is convergent for $t \sim 1$, it is suppressed for $\tau_2 \gg \log N/2\pi$, and the conclusion is that the main contribution comes actually from $\tau_2 \sim \log N/2\pi$.

Thus, finally, this region dominates since for $t \to 1$ there is no exponential suppression of the imaginary part. Rather, the imaginary part will behave as a power of N.

In order to find that power we have to have a closer look at the region where $\tau_2 \to \infty$ and $z = 1/2 + \zeta$ with ζ small. In this region we can use an approximation for the theta functions

$$\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)} = \frac{\cos(\pi\zeta)}{\pi} \left(1 + 4\cos^2(\pi\zeta)\mathrm{e}^{i2\pi\tau}\right) \tag{4.11}$$

up to terms further exponentially suppressed for $\exp(-2\pi\tau_2) \to 0$. In the same approximation we get

$$\sqrt{Q}\frac{\pi}{\tau_2}\mathrm{e}^{i2\pi\phi} = -\frac{\pi^2}{\cos^2(\pi\zeta)} - 8\pi^2\cos(2\pi\zeta)\mathrm{e}^{i2\pi\tau} + \frac{\pi}{\tau_2}\,. \tag{4.12}$$

This relation determines the values τ_1^* and ϕ^* at $\zeta = 0$, as functions of $\sqrt{Q}$ and τ_2, in the region where both go to infinity. In particular, for $\tau_2 \to \infty$ we get $\phi^* \to 1/2$. We want to investigate the behaviour of our integrand $t(\zeta, \tau)$ as a function of τ_1 and ϕ around τ_1^* and ϕ^* keeping fixed, and large, $\sqrt{Q}$ and τ_2. We have thus to re-express the complex variable ζ as a function of $\delta\tau_1 \equiv \tau_1 - \tau_1^*$ and $\delta\phi \equiv \phi - \phi^*$. From the previous equation we find (neglecting further suppressed terms):

$$\sin^2(\pi\zeta) = -i2\pi\left(\delta\phi\frac{\pi\sqrt{Q}}{\tau_2}\mathrm{e}^{i2\pi\phi^*} + \delta\tau_1 8\pi^2\mathrm{e}^{i2\pi\tau_1^*}\mathrm{e}^{-2\pi\tau_2}\right). \tag{4.13}$$

We notice that the dependence on $\delta\tau_1$ is exponentially suppressed everywhere, both here and in the expression for the integration function $t(\zeta,\tau)$. Therefore, the direction along τ_1 is almost flat: $t^{2N} \sim \exp\left(-cNe^{-2\pi\tau_2}|\delta\tau_1|\right)$ and thus, in the relevant region $\tau_2 \sim \log N/2\pi$, we do not get additional powers of N from the integration over $\delta\tau_1$.

Along the other direction we will still have conical behaviour, since we get

$$\zeta^2 = i\delta\phi\frac{2\sqrt{Q}}{\tau_2} \tag{4.14}$$

from which we get that $\mathrm{Re}(\zeta^2) = 0$ and $(\mathrm{Im}\,\zeta)^2 = |\delta\phi|\frac{\sqrt{Q}}{\tau_2}$. Substituting in the approximate expression for the theta functions we get

$$t(\zeta,\tau) \sim \frac{\sqrt{Q}}{\pi\tau_2}\exp\left(-\frac{2\pi\sqrt{Q}}{\tau_2^2}|\delta\phi|\right). \tag{4.15}$$

Therefore, integrating $t(\zeta,\tau)^{2N}$ over $\delta\phi$ we get a factor $1/N$.

The imaginary part of the result comes from the inverted gaussian integration on $\delta\tau_2$ around the minimum in τ_2 of t^{2N} (as we said, for large Q and $z = 1/2$, this minimum occurs at the (large) value of τ_2 corresponding to $\phi = \tau_1 = 1/2$). Also in this case, we do not get additional powers of N since this minimum is shallow: $t^{2N} \sim \exp\left(+cN\mathrm{e}^{-2\pi\tau_2}(\delta\tau_2)^2\right)$.

4.4 Conclusions

Finally, inserting in eq. (4.3) the contribution of this "saddle point", we get an imaginary part of ΔM^2 proportional to $N^{1/2}$, times some possible negative powers of $\log N$ coming from the final integration over Q, whose precise value depends on the details of the behaviour in Q of the slopes of the saddle point.

It is interesting to note that although restricting *ab initio* the integration to the point $z = 1/2$ we would not expect any imaginary part in the result, since for $\tau_2 \to \infty$ we would get in this case a convergent integral of a real positive quantity, still this point on the torus enters into the game as, in a sense, the analytic continuation of the point $z = 1/2 + \tau/2$, for which the integral is divergent and the imaginary part is expected.

In conclusion, we have found that the lifetime of the massive state of maximal angular momentum behaves in the limit of large mass as a constant times $\log N$ corrections. The dominant contribution comes from the neighbourhood of the point $z = 1/2$: this means that the two vertex insertions of the massive state on the torus remain fixed at fixed positions on the torus even in the limit $\tau_2 \to \infty$ where the torus degenerates. Therefore, the dominant configuration corresponds to a very long, and thus thin, handle attached to the two vertices, which are joined also by the finite part of the torus. The dominant string states on the long handle are the light ones.

JHEP11(2000)025

Therefore the physical conclusion is that the dominant two-body decay mode of this state is asymmetrical: it resembles a radiative process where the very massive state decays into another very massive state of lower mass by the emission of some massless states. Instead the fission-like processes, where the very massive state would split into two more symmetrical fragments, maybe also of very small or even zero mass (as we have explicitly computed in section 4.3), is exponentially suppressed.

5. Discussion

The result that the massive state decays by emitting low-mass particles suggests that it might be possible to interpret it as a black hole evaporating through Hawking radiation. After a charged black hole has stopped evaporating, the object that remains is believed to be a string theory bound state [9]–[12]. The evidence for this has mainly come from comparing the geometric Beckenstein–Hawking entropy to the string theory degeneracy of various D-brane configurations, and finding perfect agreement [15]. Indeed, according to the Correspondence Principle of Horowitz and Polchinski [8], when the size of the horizon drops below the size of the string the typical black hole state becomes a typical string theory bound state. It is therefore interesting to compare the massive state to a black hole at correspondence radius:

A massive state with mass $M_0 \sim \sqrt{N}$ can decay into a massive state with $M_1 \sim \sqrt{N-n}$ (N large, n finite) by emitting a massless particle of energy $\sim 1/\sqrt{N}$. As the probability per unit time of this happening behaves as $\Gamma \sim 1$, we get the radiation rate

$$\frac{dE}{dt} \sim \frac{1}{M} \,. \tag{5.1}$$

Note, however, that though it is clear that the dominantly emitted particles are massless, it does not follow directly from the above analysis that they were also soft. However, a more detailed comparison with field theory reported in appendix A indicates that the dominant radiation is soft, although one gets actually a bound on n, that is $n/\sqrt{N} \to 0$. One can also analyse the decay to a massless state by a direct computation: a preliminary investigation indicates that the radiation is soft, with n finite. We hope to return to this point in a future publication.

The length scale R_J pertinent to the particular string theory state we are studying $|\Psi_J\rangle$, for the maximal J, can be estimated by calculating the width of the state $\langle\Psi_J|X^2|\Psi_J\rangle = R_J^2 \sim M^2$. The Correspondence Principle relates the string theory state to a black hole when the length scales are comparable; the radius of the corresponding Kerr black hole [16] would therefore also be $\sim M$ and the area $A \sim M^{d-1}$. The Hawking radiation this black hole emits is that of a black body in temperature $T \sim M^{-1}$ [16], and the radiation rate is

$$\frac{dE}{dt} \sim AT^{d+1} \sim \frac{1}{M^2} \,, \tag{5.2}$$

where d is the number of space dimensions. The massive state radiates, therefore, with a stronger intensity (5.1) than a corresponding black hole. However, if one were to identify the massive state with a black hole, string theory would have to be strongly coupled, as we shall presently see, making a straight forward comparison of the radiation rates difficult.

The ADM mass is related to the radius of a black hole through $M_{\rm BH} \sim R_{\rm BH}^{d-2} g_{\rm s}^{-2}$, where $g_{\rm s}$ is the string coupling. Recall that at correspondence the length scale of the black hole is of the same order as the length scale of the quantum state $R_{\rm BH} \sim R_J$ and that in our system $R_J \sim M$. Therefore, the string coupling where the ADM mass and the mass of the string theory state grow in the same way is

$$g_{\rm s}^2 \sim N^{\frac{d-3}{2}}\,. \tag{5.3}$$

This means that we cannot reach the correspondence radius and simultaneously maintain the validity of string perturbation theory, as the calculation is performed in $d > 3$. Therefore, the massive state cannot be directly thought of as a black hole but is, really, to be described as a quantum mechanical state.

In $d+1 \leq 5$ there is an upper bound for the angular momentum of a black hole with given mass that arises from requiring that there be no naked singularities. As was noted in [17, 16], in $d+1 > 5$ there is no such bound, and all Kerr solutions posses a protecting horizon. It is, however, amusing to note that from the simple requirement that the angular momentum arise from a mass distribution still inside the horizon, we get a bound $J < R_{\rm BH} M$, which reduces to an inequality

$$g_{\rm s}^2 \gtrsim N^{\frac{d-3}{2}}\,. \tag{5.4}$$

This bound would be saturated at the correspondence radius (5.3): it seems, therefore, that in weak coupling, where we perform our calculations, a black hole interpretation is not feasible; On the other hand, when the black hole interpretation is adequate, the pertinent string theory is necessarily strongly coupled. Therefore, one could expect that varying the string coupling the system undergoes a phase transition, when the correspondence radius is reached. This suggests that the descriptions in terms of a quantum state and in terms of a black hole are *complementary*, rather than equivalent.

Acknowledgments

We would like to acknowledge collaboration with Daniele Amati in the early stages of this work, and to thank him for proposing the study of excited string states at one-loop as a tool for investigating quantum gravity effects. We thank also Matteo Bertolini and Kristian Jenssen for discussions.

JHEP11(2000)025

A. Massless radiation

In order to compare the results of sections 4.3 and 4.4 with field theory, we consider the one-loop Feynman amplitude describing the radiative correction to the propagator of a particle of square energy p^2, in a theory with a three-point vertex. For simplicity, we consider spinless particles. The starting point is the euclidean expression

$$\begin{aligned} A &= \int d^D k \frac{1}{k^2+m_a^2} \frac{1}{(p-k)^2+m_b^2} \\ &= \int_0^\infty \int_0^\infty da db \int d^D k \mathrm{e}^{-a(k^2+m_a^2)-b((p-k)^2+m_b^2)} \\ &= \pi^{D/2} \int_0^\infty \frac{d\tau_2}{\tau_2^{D/2-1}} \int_0^1 dy \mathrm{e}^{Ny(1-y)\tau_2 - y\tau_2 m_a^2 - (1-y)\tau_2 m_b^2} . \end{aligned} \tag{A.1}$$

We have made here the continuation to the Minkowski space $p^2 = -N$. Now, in the corner $\tau_2 \to \infty, y \to 0$ (cf. eq. (3.9)) the leading contribution comes from the massless case $m_b^2 = 0$, and

$$A = \pi^{D/2} \int_0^\infty \frac{d\tau_2}{\tau_2^{D/2-1}} \int_0^1 dy \mathrm{e}^{-Ny^2\tau_2 + (N-m_a^2)y\tau_2} . \tag{A.2}$$

The dominant contribution comes from $y = (N - m_a^2)/(2N)$, therefore $y = 0$ corresponds to $n/N \to 0$ (taking $m_a^2 = N - n$).

Moreover, let us now expand $y = 0 + \eta$ and compare the result with eq. (4.15). Remember that $\sqrt{Q}/\tau_2 \to 1$ and that $|\delta\phi| = (\mathrm{Im}\,\zeta)^2$. Then taking $\mathrm{Im}\,\zeta = \eta\tau_2$ we see

$$t^N \sim \mathrm{e}^{-2\pi N\eta^2\tau_2} . \tag{A.3}$$

Since $\tau_2 \sim \ln N$ the relevant range is $\eta \sim 1/\sqrt{N}$ neglecting $\ln N$ corrections. Furthermore, in eq. (4.15) there is no linear term in η in the exponent; it matches eq. (A.2) if $n/\sqrt{N} \to 0$. This implies that the energy of the massless particle is soft.

References

[1] D.J. Gross and P.F. Mende, *String theory beyond the Planck scale*, *Nucl. Phys.* **B 303** (1988) 407.

[2] V. Balasubramanian and I.R. Klebanov, *Some aspects of massive world-brane dynamics*, *Mod. Phys. Lett.* **A 11** (1996) 2271 [hep-th/9605174].

[3] B. Sundborg, *Selfenergies of massive strings*, *Nucl. Phys.* **B 319** (1989) 415.

[4] T. Damour and G. Veneziano, *Self-gravitating fundamental strings and black holes*, *Nucl. Phys.* **B 568** (2000) 93 [hep-th/9907030].

[5] D. Amati, M. Ciafaloni and G. Veneziano, *Superstring collisions at planckian energies*, *Phys. Lett.* **B 197** (1987) 81.

[6] A. Feinstein, K.E. Kunze and M.A. Vazquez-Mozo, *Initial conditions and the structure of the singularity in pre-big-bang cosmology*, *Class. and Quant. Grav.* **17** (2000) 3599 [hep-th/0002070].

[7] V. Bozza and G. Veneziano, *O(d,d) invariant collapse/inflation from colliding superstring waves*, *J. High Energy Phys.* **10** (2000) 035 [hep-th/0007159].

[8] G.T. Horowitz and J. Polchinski, *A correspondence principle for black holes and strings*, *Phys. Rev.* **D 55** (1997) 6189 [hep-th/9612146].

[9] G. Veneziano, *A stringy nature needs just two constants*, *Europhys. Lett.* **2** (1986) 199.

[10] M.J. Bowick, L. Smolin and L.C. Wijewardhana, *Does string theory solve the puzzles of black hole evaporation?*, *Gen. Rel. Grav.* **19** (1987) 113.

[11] G. 't Hooft, *The black hole interpretation of string theory*, *Nucl. Phys.* **B 335** (1990) 138.

[12] L. Susskind, *Some speculations about black hole entropy in string theory*, hep-th/9309145.

[13] J.H. Schwarz, *Superstring theory*, *Phys. Rep.* **89** (1982) 223.

[14] G. Szegö, *Orthogonal polynomials*, AMS Colloqium publications Vol. XXIII AMS, Providence, 1939.

[15] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, *Phys. Lett.* **B 379** (1996) 99 [hep-th/9601029].

[16] G.T. Horowitz and A. Sen, *Rotating black holes which saturate a Bogomol'nyi bound*, *Phys. Rev.* **D 53** (1996) 808 [hep-th/9509108].

[17] R.C. Myers and M.J. Perry, *Black holes in higher dimensional space-times*, *Ann. Phys. (NY)* **172** (1986) 304.